Properties of Expectations of Functions of Martingale Diffusions

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Abstract

Given a real valued and time-inhomogeneous martingale diffusion X, we investigate the properties of functions defined by the conditional expectation $f(t, X_t) = \mathbb{E}\left[g(X_T)|\mathcal{F}_t\right]$. We show that whenever g is monotonic or Lipschitz continuous then $f(t, x)$ will also be monotonic or Lipschitz continuous in x. If g is convex then $f(t, x)$ will be convex in x and decreasing in t. We also define the *marginal support* of a process and show that it almost surely contains the paths of the process. Although f need not be jointly continuous, we show that it will be continuous on the marginal support of X. We prove these results for a generalization of diffusion processes that we call 'almost-continuous diffusions', and includes all continuous and strong Markov processes.

1 Introduction

Suppose that we have a real valued Markov process X, and any times $T > t \geq 0$. Then, the Markov property says that for any measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $g(X_T)$ is integrable, there is a function f satisfying

$$(1) \qquad f(t, X_t) = \mathbb{E}\left[g(X_T)|\mathcal{F}_t\right]$$

for every $t \leq T$. The aim of this paper is to show that various properties of f follow from the properties of g. When X is a time inhomogeneous diffusion (i.e., strong Markov and continuous) then it is shown that $f(t, x)$ is increasing in x whenever g is increasing. If X is also a martingale we show that $f(t, x)$ is Lipschitz continuous in x whenever $g(x)$ is Lipschitz continuous. If, furthermore, g is convex then $f(t, x)$ will be both convex in x and decreasing in t. The proof of these results makes use of the coupling methods used by [4].

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We also look at the continuity of f. In general, it may not be possible to choose $f(t,x)$ to be jointly continuous everywhere. However, we define the *marginal support* of a process, which is a Borel subset of $\mathbb{R}_+ \times \mathbb{R}$ defined in terms of the supports of the marginal distributions, and show that it almost-surely contains the paths of the process and that f is jointly continuous on this set.

All of these results are proven for a generalization of diffusion processes that we refer to as *almost-continuous* diffusions. Such processes are defined by the strong Markov property and a pathwise almost-continuity condition, and we also characterize the almost-continuous martingales in terms of the jumps of the process and its marginal support.

The situation where g is convex and monotonic occurs frequently in mathematical finance. In this case, $g(X_T) = (X_T - k)_+$ is the payoff of a call option with strike k and maturity T, and $f(t, X_t)$ is its value at time t as a function of the underlying asset price X. It is well known that under many situations the function $f(t,x)$ would then be convex and increasing in x and decreasing in t. For example, this is a property of the Black-Scholes pricing formula, or indeed any model in which X is the exponential of a process with independent increments (see, for example, [7]).

Another case that has been studied by several authors is when X is a diffusion described by an SDE of the form

$$(2) \qquad\qquad\qquad dX_t = \sigma(t, X_t)\, dW_t$$

where W is a Brownian motion. This situation has been investigated by [1, 4, 6]. In [1] they differentiate the following PDE for f,

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x) = 0.$$

Then it is shown that the derivative of f is increasing with respect to x. In [6] they use a method of stochastic flows. Both these approaches require σ to be continuously differentiable. In [5] the notion of volatility time is used to show that $f(t,x)$ is decreasing in t, and then deduce convexity in x from the PDE above. In [4] a simple coupling method is used to prove convexity in x, which doesn't place any restrictions on σ, other than those necessary for the SDE (2) to have unique solutions that are martingales. We employ these coupling techniques in the current paper in order to prove the monotonicity, Lipschitz continuity and convexity results for f. However, rather than assuming that X is the unique solution of an SDE such as (2) we instead work with the more general definition of a diffusion as a continuous

and strong Markov process, and further generalize to the following class of processes. Except where explicitly stated otherwise, we assume that all stochastic processes are defined with respect to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$.

Definition 1.1. *Let X be a real valued stochastic process. Then,*

1. *X is* strong Markov *if for every bounded, measurable $g : \mathbb{R} \to \mathbb{R}$ and every $t \in \mathbb{R}_+$ there exists a measurable $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that*

$$f(\tau, X_\tau) = \mathbb{E}\left[g(X_{\tau+t}) \mid \mathcal{F}_\tau \right]$$

for every finite stopping time τ.

2. *X is* almost-continuous *if it is cadlag, continuous in probability and given any two independent, identically distributed cadlag processes Y, Z with the same distribution as X and for every $s < t \in \mathbb{R}_+$ we have*

$$\mathbb{P}\left(Y_s < Z_s, Y_t > Z_t \text{ and } Y_u \neq Z_u \text{ for every } u \in (s, t) \right) = 0$$

3. *X is an* almost-continuous diffusion *if it is strong Markov and almost-continuous.*

The second condition above is equivalent to saying that $Y - Z$ cannot change sign without hitting zero and, by the intermediate value theorem, is clearly true for continuous processes. Examples of almost-continuous diffusions that are not continuous include, for example, Poisson and compensated Poisson processes. Also, almost-continuous diffusions arise as limits of continuous diffusions under the topology of convergence of finite-dimensional distributions (as we shall show in a future paper). We will often abbreviate 'almost-continuous diffusion' to ACD.

To further see what the almost-continuous property means, we can look at an alternative way of defining it. First, given any real valued random variable X then its support, $\text{Supp}(X)$, is defined to be the smallest closed subset of $\mathbb{R}$ such that $\mathbb{P}(X \in \text{Supp}(X)) = 1$. We define the *marginal support* of a process X in terms of the supports of its marginal distributions.

Definition 1.2. *Let X be a real valued stochastic process. Then, its* marginal support *is*

$$\text{MSupp}(X) = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : x \in \text{Supp}(X_t)\} .$$

If X is right-continuous in probability then, as we shall show, its marginal support will be a Borel measurable subset of $\mathbb{R}_+ \times \mathbb{R}$. We now state the alternative definition of almost-continuous martingales.

Lemma 1.3. *Let X be a real valued cadlag martingale which is continuous in probability. Then it is almost-continuous if and only if*

$$(3) \qquad \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : X_{t-} < x < X_t \text{ or } X_t < x < X_{t-}\}$$

is almost surely disjoint from the marginal support of X.

So a martingale is almost-continuous if it cannot jump past any points in its marginal support. The proof of this is rather complicated, and is left until Section 4.

We now state the result that $f(t,x)$ satisfying (1) can be chosen to be increasing in x whenever g is increasing.

Theorem 1.4. *Let X be an almost-continuous diffusion. Choose any increasing function $g : \mathbb{R} \to \mathbb{R}$ and $T \in \mathbb{R}_+$ such that $\mathbb{E}\left[|g(X_T)|\right] < \infty$.*

Then, for every $t \in [0,T]$ there exists an increasing function $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ such that $f(X_t) = \mathbb{E}\left[g(X_T)|\mathcal{F}_t\right]$.

The proof of this is left until Section 3. Note that this applies to all almost-continuous diffusions, and not just martingales. Under the additional assumption that X is a martingale then by a simple corollary of Theorem 1.3, f will be Lipschitz continuous in x whenever g is. Recall that a function f on the real numbers is Lipschitz continuous if there exists real numbers $k \leq K$ such that

$$(4) \qquad k(y-x) \leq f(y) - f(x) \leq K(y-x)$$

for all $x < y \in \mathbb{R}$. This is satisfied if and only if the derivative $df(x)/dx$ exists in the measure theoretic sense and can be chosen such that $k \leq df(x)/dx \leq K$ for all $x \in \mathbb{R}$. So, we shall write $k \leq f' \leq K$ to mean that f satisfies inequality (4) for all $x < y \in \mathbb{R}$.

Theorem 1.5. *Let X be an ACD martingale, and $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $k \leq g' \leq K$. Then for any $t < T \in \mathbb{R}_+$ there exists a Lipschitz continuous $f : \mathbb{R} \to \mathbb{R}$ such that $k \leq f' \leq K$ and $f(X_t) = \mathbb{E}\left[g(X_T)|\mathcal{F}_t\right]$.*

Proof. As X is a martingale and $k \leq g' \leq K$,

$$\mathbb{E}\left[|g(X_T)|\right] \leq |g(0)| + \max\left(-k, K\right)\mathbb{E}\left[|X_T|\right] < \infty,$$

As X is also Markov, we can define f by $f(X_t) = \mathbb{E}\left[g(X_T)|\mathcal{F}_t\right]$. Then, the martingale property gives

$$f(X_t) - kX_t = \mathbb{E}\left[g(X_T) - kX_T|\mathcal{F}_t\right],$$
$$KX_t - f(X_t) = \mathbb{E}\left[KX_T - g(X_T)|\mathcal{F}_t\right].$$

As $k \leq g' \leq K$ we see that $g(x) - kx$ and $Kx - g(x)$ are increasing, so Theorem 1.4 tells us that $f(x) - kx$ and $Kx - f(x)$ are increasing on some set $S \subseteq \mathbb{R}$ with $\mathbb{P}\left(X_t \in S\right) = 1$. Therefore equation (4) is satisfied for all $x < y \in S$. Finally, f extends to the closure of S by uniform continuity, and can be linearly interpolated and extrapolated outside of the support of X_t. $\qquad\square$

We now state the properties of $f(t, x)$ satisfying (1) when g is convex. This is the case investigated in [1, 4, 6] and shows that whenever the price of a financial asset is modeled by a continuous (or almost-continuous) and strong Markov process then option prices will be convex in the asset price and decreasing in time.

Theorem 1.6. *Let X be an ACD martingale, $g : \mathbb{R} \to \mathbb{R}$ be a convex and Lipschitz continuous function and choose any $T \in \mathbb{R}_+$. Then there exists a function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $f(t, x)$ is convex in x, decreasing in t and*

$$(5) \qquad\qquad f(t, X_t) = \mathbb{E}\left[g(X_T)|\mathcal{F}_t\right]$$

for every $t \in [0, T]$.

The proof of this is given in Section 3, and follows the coupling idea of [4]. As we now show, f will be jointly continuous on the marginal support of X. The requirement in the following result that locally $f(t, x)$ is uniformly continuous in x, just means that it is uniformly continuous in x on every bounded subset of $[0, T] \times \mathbb{R}$.

Theorem 1.7. *Let X be a real valued and adapted process which is continuous in probability. Suppose that $T > 0$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that locally $f(t, x)$ is uniformly continuous in x and $f(t, X_t)$ is a martingale. Then, f is jointly continuous on the marginal support of X.*

Proof. Set $M_t = f(t, X_t)$ for $t \leq T$. As it is a martingale, it has left and right limits everywhere. We shall prove left and right continuity separately. First

choose any sequence $t_n \in [0, T]$ with $t_n \downarrow\downarrow t$. The continuity in probability of X_t and the local uniform continuity of $f(t, x)$ in x gives

$$M_{t+} = \lim_{n\to\infty} f(t_n, X_{t_n}) = \lim_{n\to\infty} f(t_n, X_t)$$

where convergence is in probability. In particular, we see that M_{t+} is $\mathcal{F}_t$-measurable, so $M_t = \mathbb{E}[M_{t+}|\mathcal{F}_t] = M_{t+}$ and

$$f(t, X_t) = M_t = \lim_{n\to\infty} f(t_n, X_t).$$

Then the local uniform continuity of $f(t, x)$ in x shows that $f(t_n, x) \to f(t, x)$ for all x in the support of X_t, and $f(t, x)$ is indeed right-continuous in t on $\mathrm{MSupp}(X)$.

Now choose $t_n \in [0, T]$ such that $t_n \uparrow\uparrow t$. Arguing as above, we have

$$M_{t-} = \lim_{n\to\infty} f(t_n, X_{t_n}) = \lim_{n\to\infty} f(t_n, X_t).$$

However, the continuity in probability of X_t implies that X_t is $\mathcal{F}_{t-}$-measurable. So, $M_{t-} = \mathbb{E}[M_t|\mathcal{F}_{t-}] = M_t$ and,

$$f(t, X_t) = M_t = \lim_{n\to\infty} f(t_n, X_t).$$

As above, this implies that $f(t_n, x) \to f(t, x)$ for every x in the support of X_t, and $f(t, x)$ is left-continuous in t on the marginal support of X. □

In particular, suppose that X is an ACD martingale, g is Lipschitz continuous, and f is defined by equation (1). Then $f(t, X_t)$ is a martingale and Theorem 1.5 says that $f(t, x)$ can be chosen to be Lipschitz continuous in x. The above result then states that f is jointly continuous on $\mathrm{MSupp}(X)$. As we shall show in Section 4, the paths of X will lie within its marginal support (Lemmas 4.3 and 4.4) at all times, so we can conclude that $f(t, X_t)$ almost surely has cadlag paths and can only jump at times when X jumps.

In fact, as we now show by a counterexample, it is generally not possible to choose f to be continuous outside of the marginal support of X. To construct our example, we first let B be a Brownian motion and let τ be the random time

$$\tau = \inf\{t \in \mathbb{R}_+ : |B_t| = 1\} < \infty.$$

This is a stopping time with respect to the filtration generated by B, and $B_\tau = \pm 1$. Define the process X by

$$X_t = \begin{cases} B_{t/(1-t)}, & \text{if } 0 \le t < \tau/(1+\tau), \\ B_\tau, & \text{if } \tau/(1+\tau) \le t \le 1, \\ B_{\tau+t-1}, & \text{if } t > 1. \end{cases}$$

Then, under its natural filtration, X is a continuous strong Markov martingale satisfying the SDE (2) with

$$\sigma(t,x) = \frac{1}{1-t}1_{\{(t,x)\in[0,1)\times(-1,1)\}} + 1_{\{t\geq 1\}}$$

The marginal support of X is

$$\mathrm{MSupp}(X) = \{(0,0),(1,-1),(1,1)\} \cup ((0,1)\times[-1,1]) \cup ((1,\infty)\times\mathbb{R}).$$

Note, in particular, that at time $t = 1$ the support of X_t is not connected, consisting of just the two points $\{-1,1\}$. Now let $g(x) = x^2$, choose any $T > 1$ and let $f(t,x)$ satisfy equation (1). If $t \in [1,T]$ then

$$f(t,X_t) = \mathbb{E}\left[X_T^2|\mathcal{F}_t\right] = X_t^2 + T - t.$$

And, for $t \leq 1$ we can use $X_1 = \pm 1$ to get

$$f(t,X_t) = \mathbb{E}\left[f(1,X_1)|\mathcal{F}_t\right] = \mathbb{E}\left[X_1^2 + T - 1|\mathcal{F}_t\right] = T$$

As $|X_t| \leq 1$ for $t < 1$, we can choose whatever value we like for $f(t,x)$ in the range $t < 1$ and $|x| > 1$. In particular, f can be chosen to be

$$f(t,x) = \begin{cases} x^2 + T - t, & \text{if } 1 \leq t \leq T, \\ T, & \text{if } t < 1 \text{ and } |x| \leq 1, \\ x^2 + T - 1, & \text{if } t < 1 \text{ and } |x| > 1. \end{cases}$$

Note that this is convex in x and right-continuous and decreasing in t. Although Theorem 1.6 does not quite apply in this case, because the derivative of g is not bounded, it should be clear that it could be approximated by Lipschitz continuous functions, so the fact that $f(t,x)$ is convex in x and decreasing in t will still follow from Theorem 1.6. We just chose $g(x) = x^2$ for simplicity, but any convex and non-linear function would do just as well.

We see that f is jointly continuous everywhere except for the line $\{1\} \times (-1,1)$, on which

$$f(1,x) = x^2 + T - 1,$$
$$f(1-,x) = T.$$

So f is discontinuous on this line, but is continuous everywhere on the marginal support of X.

2   The Strong Markov Property

In order to prove the results of Section 1 we shall make use of the strong Markov property. However, the form in which it is stated in Definition 1.1 is not the easiest to work with, so we shall make use of the following lemma which rephrases the strong Markov property in a slightly different way. We are only interested in real valued processes here, although the following proofs will generalize to any Polish space.

Lemma 2.1. *Let X be a real valued cadlag process that satisfies the strong Markov property. For every $T \in \mathbb{R}_+$ and bounded measurable $g : \mathbb{R} \to \mathbb{R}$ there exists a measurable $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that for every stopping time τ*

$$1_{\{\tau \le T\}} f(\tau, X_\tau) = 1_{\{\tau \le T\}} \mathbb{E}\left[g(X_T) \mid \mathcal{F}_\tau\right].$$

Proof. Choose any $T \in \mathbb{R}_+$ and bounded continuous $g : \mathbb{R} \to \mathbb{R}$. From Definition 1.1 of the strong Markov property, for every $\alpha > 0$ there exists a measurable $f_\alpha : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that for every stopping time τ,

$$1_{\{T < \infty\}} f_\alpha(\tau, X_\tau) = 1_{\{\tau < \infty\}} \mathbb{E}\left[g(X_{\tau+\alpha}) \mid \mathcal{F}_\tau\right].$$

Now pick any $n \in \mathbb{N}$. For any stopping time τ let τ_n be the stopping time

$$\tau_n = \begin{cases} \infty, & \text{if } \tau > T, \\ \tau + ([n(T - \tau)] + 1)/n, & \text{if } \tau \le T. \end{cases}$$

Here we are using the notation $[x]$ to denote the largest integer that is less than or equal to x. Define the measurable function $h_n : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$h_n(t, x) = \sum_{k=1}^{\infty} 1_{\left\{T - \frac{k}{n} < t \le T - \frac{k-1}{n}\right\}} f_{\frac{k}{n}}(t, x).$$

So,

$$
\begin{aligned}
1_{\{\tau \le T\}} h_n(\tau, X_\tau) &= \sum_{k=1}^{\infty} 1_{\left\{t - \frac{k}{n} < \tau \le T - \frac{k-1}{n}\right\}} \mathbb{E}\left[g\left(X_{\tau + \frac{k}{n}}\right) \middle| \mathcal{F}_\tau\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{\infty} 1_{\left\{T - \frac{k}{n} < \tau \le T - \frac{k-1}{n}\right\}} g\left(X_{\tau + \frac{k}{n}}\right) \middle| \mathcal{F}_\tau\right] \\
&= 1_{\{\tau \le T\}} \mathbb{E}\left[g(X_{\tau_n}) \middle| \mathcal{F}_\tau\right].
\end{aligned}
$$

From the definition of τ_n we have $T < \tau_n \leq T + \frac{1}{n}$ whenever $\tau \leq T$, so $\tau_n \to T$ as $n \to \infty$. By the right-continuity of X and the continuity of g, we can use bounded convergence to get

$$1_{\{\tau \leq T\}} \mathbb{E}\left[g(X_T)|\mathcal{F}_\tau\right] = \lim_{n \to \infty} 1_{\{\tau \leq T\}} h_n(\tau, X_\tau).$$

So the result for continuous g follows by setting

$$f(t, x) = \limsup_{n \to \infty} h_n(t, x).$$

Then it extends to arbitrary bounded measurable g by the Monotone Class Lemma. $\qquad\qquad\square$

Given a strong Markov process X we shall require the existence of a process which is independent of X and with the same distribution. One way to construct such a process is to take the product of the underlying probability space with itself.

For any two filtered probability spaces

$$\left(\Omega_1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t \in \mathbb{R}_+}, \mathbb{P}_1\right), \ \left(\Omega_2, \mathcal{F}^2, (\mathcal{F}_t^2)_{t \in \mathbb{R}_+}, \mathbb{P}_2\right)$$

we can form the product

$$(6) \quad \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P}\right) = \left(\Omega_1 \times \Omega_2, \mathcal{F}^1 \otimes \mathcal{F}^2, (\mathcal{F}_t^1 \otimes \mathcal{F}_t^2)_{t \in \mathbb{R}_+}, \mathbb{P}_1 \otimes \mathbb{P}_2\right).$$

Any process defined on one of the two original probability spaces lifts to a process defined on their product (by combining with the projection map). When we pass to a larger probability space like this, we need to know that strong Markov processes will still be strong Markov.

Lemma 2.2. *Let*

$$X : \mathbb{R}_+ \times \Omega_1 \to \mathbb{R},$$
$$(t, \omega_1) \mapsto X_t(\omega_1)$$

be a strong Markov process on the filtered probability space $\left(\Omega_1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t \in \mathbb{R}_+}, \mathbb{P}_1\right)$.

If we lift X to the process $\tilde{X}$ defined on the product of the filtered probability spaces in equation (6)

$$\tilde{X} : \mathbb{R}_+ \times \Omega \to \mathbb{R},$$
$$(t, \omega_1, \omega_2) \mapsto \tilde{X}_t(\omega_1, \omega_2) \equiv X_t(\omega_1)$$

then $\tilde{X}$ is also strong Markov.

Proof. Choose any bounded measurable $g : \mathbb{R} \to \mathbb{R}$ and $\alpha > 0$. As X is strong Markov there exists a measurable $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that for every finite stopping time τ defined on the first filtered probability space,

$$1_{\{\tau < \infty\}} f(\tau, X_\tau) = 1_{\{\tau < \infty\}} \mathbb{E}\left[g(X_{\tau+\alpha})|\mathcal{F}_\tau^1\right].$$

This is equivalent to the statement

$$\mathbb{E}\left[U\left(f(\tau, X_\tau) - g(X_{\tau+\alpha})\right)\right] = 0$$

for every bounded $\mathcal{F}_\tau^1$-measurable random variable U.

Now choose any finite stopping time τ defined with respect to the product of the filtered probability spaces, and let U be a bounded $\mathcal{F}_\tau$-measurable random variable. For any $\omega_2 \in \Omega_2$ define the $\mathcal{F}^1$-stopping time

$$\tau_{\omega_2} : \Omega_1 \to \mathbb{R}_+,$$
$$\tau_{\omega_2}(\omega_1) = \tau(\omega_1, \omega_2)$$

and the $\mathcal{F}_{\tau_{\omega_2}}^1$-measurable random variable

$$U_{\omega_2} : \Omega_1 \to \mathbb{R},$$
$$U_{\omega_2}(\omega_1) = U(\omega_1, \omega_2).$$

Then,

$$\mathbb{E}\left[U\left(g(\tau, \tilde{X}_\tau) - f(\tilde{X}_{\tau+\alpha})\right)\right]$$
$$= \int \mathbb{E}\left[U_{\omega_2}\left(g(\tau_{\omega_2}, X_{\tau_{\omega_2}}) - f(X_{\tau_{\omega_2}+\alpha})\right)\right] d\mathbb{P}(\omega_2)$$
$$= 0$$

so $\tilde{X}$ is indeed strong Markov.   □

3   Conditional Expectations

We now move on to proving theorems 1.4 and 1.6. The method of proof employs a similar idea to the coupling arguments used in [4]. The idea is to take several independent copies of the process, and observe them up until the first time that they touch.

If X is any strong Markov process, $T \in \mathbb{R}_+$ and $g : \mathbb{R} \to \mathbb{R}$ is any measurable function satisfying $\mathbb{E}\left[|g(X_T)|\right] < \infty$, then it follows from Lemma

2.1 that there exists a measurable $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $h(T, x) = g(x)$ and

$$(7) \qquad 1_{\{\tau > t\}} h(t, X_t) = 1_{\{\tau > t\}} \mathbb{E} \left[h(T \wedge \tau, X_{T \wedge \tau}) | \mathcal{F}_t \right].$$

for any $t \in [0, T]$ and any stopping time τ. We can make use of this to prove the results we need for conditional expectations of almost-continuous diffusions. We start by proving Theorem 1.4, that conditional expectations of increasing functions are themselves increasing.

Proof of Theorem 1.4. First we can extend the probability space by taking the product of the underlying filtered probability space with itself. Then there exists a process Y independent of X which is also strong Markov and has the same distribution as X (from Lemma 2.2). As X is strong Markov there exists a measurable $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $h(T, x) = g(x)$ and equation (7) is satisfied for any stopping time τ. Setting $f(x) = h(t, x)$ gives $f(X_t) = \mathbb{E}[g(X_T)|\mathcal{F}_t]$. We just need to show that f can be chosen to be increasing.

Let τ be the stopping time

$$\tau = \inf \left\{ s \in [t, \infty) : X_s \geq Y_s \right\}.$$

Note that $\{\tau > t\} = \{X_t < Y_t\}$.

As X is almost-continuous, $X_\tau = Y_\tau$ whenever $t < \tau < \infty$. Also, as g is increasing, $g(X_T) \leq g(Y_T)$ whenever $\tau > T$. So equation (7) gives

$$1_{\{\tau > t\}} (f(Y_t) - f(X_t)) = 1_{\{\tau > t\}} \mathbb{E} \left[h(T \wedge \tau, Y_{T \wedge \tau}) - h(T \wedge \tau, X_{T \wedge \tau}) \big| \mathcal{F}_t \right]$$
$$= 1_{\{\tau > t\}} \mathbb{E} \left[1_{\{\tau > T\}} (g(Y_T) - g(X_T)) \big| \mathcal{F}_t \right] \geq 0.$$

Therefore, $f(X_t) \leq f(Y_t)$ whenever $X_t < Y_t$ (a.s.).

This shows that f is increasing in an almost sure sense. That is, there exists a measurable subset S of $\mathbb{R}$ such that f is increasing on S and $\mathbb{P}(X_t \in S) = 1$. So, we can extend f outside of S by

$$f(x) = \sup \{ f(y) : y \in (-\infty, x] \cap S \} \in \mathbb{R} \cup \{\pm\infty\}. \qquad \square$$

We now show that convexity is also preserved under taking conditional expectations of functions of ACD martingales. This was proven in [4] for the case of continuous diffusions driven by an SDE, by considering running three independent copies of the process and using a coupling method. However, their proof only really relied on continuity and the strong Markov condition, and can be extended to all ACD martingales.

We use the condition that a function $f : S \to \mathbb{R}$ for $S \subseteq \mathbb{R}$ is convex if and only if

$$(8) \qquad (z - y)f(x) + (y - x)f(z) + (x - z)f(y) \geq 0$$

for any $x < y < z \in S$. Also, for any convex function f from the reals to the reals, the right hand derivative $f'(x)$ exists and is right-continuous and increasing in x.

Lemma 3.1. *Let X be an ACD martingale, and $g : \mathbb{R} \to \mathbb{R}$ be a convex function with $k \leq g' \leq K$. Then for any $t < T \in \mathbb{R}_+$ there exists a convex function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$(9) \qquad f(X_t) = \mathbb{E}\left[g(X_T)|\mathcal{F}_t\right]$$

and $k \leq f' \leq K$.

Proof. As $k \leq g' \leq K$, g is Lipschitz continuous. By Theorem 1.5 there exists a Lipschitz continuous $f : \mathbb{R} \to \mathbb{R}$ with $k \leq f' \leq K$ and satisfying equation (9).

We can extend the probability space by taking the product of three copies of the underlying filtered probability space. Then, by Lemma 2.2, there exists independent strong Markov processes X^1, X^2, X^3 each of which have the same distribution as X. As the X^i are strong Markov there exists a measurable $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $h(T, x) = g(x)$ and equation (7) is satisfied for any stopping time τ, and with X^i in place of X ($i = 1, 2, 3$). Then,

$$f(X_t) = h(t, X_t) \text{ (a.s.)}.$$

We now define the process

$$M_s = (X_s^3 - X_s^2)h(s, X_s^1) + (X_s^2 - X_s^1)h(s, X_s^3) + (X_s^1 - X_s^3)h(s, X_s^2)$$

and the stopping time

$$\tau = \inf\left\{s \in [t, \infty) : X_s^1 \geq X_s^2 \text{ or } X_s^2 \geq X_s^3\right\}.$$

In particular, note that $\{\tau > t\} = \{X_t^1 < X_t^2 < X_t^3\}$.

As the X^i are cadlag martingales adapted to independent filtrations, equation (7) gives

$$1_{\{\tau > t\}} X_t^i f(X_t^j) = 1_{\{\tau > t\}} \mathbb{E}\left[X_{T \wedge \tau}^i h(T \wedge \tau, X_{T \wedge \tau}^j)\big|\mathcal{F}_t\right]$$

whenever $i \neq j$. So, as M is a linear combination of such terms,

$$1_{\{\tau > t\}} M_t = 1_{\{\tau > t\}} \mathbb{E}\left[M_{T \wedge \tau} | \mathcal{F}_t\right].$$

Now, if $t < \tau \leq T$ then either $X_\tau^1 = X_\tau^2$ or $X_\tau^3 = X_\tau^2$ (because X is almost-continuous). So, $M_\tau = 0$. On the other hand, if $\tau > T$ then $X_T^1 < X_T^2 < X_T^3$, so the convexity of g gives $M_T \geq 0$.

In either case this says that $M_{T \wedge \tau} \geq 0$ whenever $\tau > t$, so $1_{\{\tau > t\}} M_t \geq 0$ or, equivalently,

$$(X_t^3 - X_t^2)f(X_t^1) + (X_t^2 - X_t^1)f(X_t^3) + (X_t^1 - X_t^3)f(X_t^2) \geq 0$$

whenever $X_t^1 < X_t^2 < X_t^3$ (a.s.). As f is continuous, this shows that it is convex on the support of X_t.

We can extend f to any bounded open interval in the complement of $\mathrm{Supp}(X_t)$ by linear interpolation. If $\mathrm{Supp}(X_t)$ is uniformly bounded above then we can extrapolate f linearly with gradient K above the support of X_t. Similarly, if $\mathrm{Supp}(X_t)$ is bounded below then we extrapolate f linearly with gradient k below the support of X_t, which gives a convex function. $\square$

We can extend the previous result to prove that conditional expectations of convex functions give convex functions that are decreasing in time, as was stated in Theorem 1.6.

Proof of Theorem 1.6. As g is Lipschitz continuous, there exist real numbers k, K such that $k \leq g' \leq K$. By Lemma 3.1 there is a function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfying equation (5) and such that $f(t, x)$ is convex and Lipschitz-continuous in x with derivative satisfying $k \leq f_{,2} \leq K$. By linear interpolation and extrapolation outside the support of X_t, we can suppose that $f(t, x)$ is linear in X across each of the connected components of $\mathbb{R} \setminus \mathrm{Supp}(X_t)$. Furthermore we can assume that $f_{,2}(t, x)$ is equal to K whenever x is an upper bound for $\mathrm{Supp}(X_t)$ and k whenever it is a lower bound.

It only needs to be shown that $f(t, x)$ is decreasing in t. For any $s < t \in [0, T]$ we can apply Jensen's inequality,

$$f(s, X_s) = \mathbb{E}\left[f(t, X_t) | \mathcal{F}_s\right] \geq f(t, X_s).$$

So $f(s, x) \geq f(t, x)$ for every x in the support of X_s. This inequality extends to the bounded open components of $\mathbb{R} \setminus \mathrm{Supp}(X_s)$ as we chose $f(s, x)$ to be linear across these intervals. So, it holds for every $x \in [a, b]$ where a is the

infimum and b is the supremum of $\mathrm{Supp}(X_s)$. If $b < \infty$ and $x > b$ the inequality $f_{,2}(s,x) = K \geq f_{,2}(t,x)$ gives

$$f(s,x) = f(s,b) + K(x-b) \geq f(t,b) + K(x-b) \geq f(t,x).$$

Similarly, if $a > -\infty$ and $x < a$ the inequality $f_{,2}(s,x) = k \leq f_{,2}(t,x)$ gives

$$f(s,x) = f(s,a) + k(x-a) \geq f(t,a) + k(x-a) \geq f(t,x)$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4   Marginal Supports

In this section we shall prove a few results concerning the marginal support (Definition 1.2) of almost-continuous processes. In particular, we show that the paths of a process are contained in its marginal support (Lemma 4.3), and prove Lemma 1.3 characterizing almost-continuous martingales in terms of their marginal support.

We start by showing that the marginal support is always Borel measurable.

Lemma 4.1. *Let X be a real valued stochastic process that is right-continuous in probability. For any real numbers $a < b$ set*

$$S_{a,b} = \left\{ t \in \mathbb{R}_+ : (a,b) \cap \mathrm{Supp}(X_t) = \emptyset \right\}.$$

Then $S_{a,b}$ is Borel measurable and

(10) $$\mathrm{MSupp}(X) = \mathbb{R}_+ \times \mathbb{R} \setminus \bigcup \left\{ S_{a,b} \times (a,b) : a < b \in \mathbb{Q} \right\}.$$

Proof. For any real numbers $a < b$ define $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(t) = \mathbb{E}\left[\min\left((X_t - a)_+, (b - X_t)_+ \right) \right].$$

The right-continuity of X implies that f is also right-continuous, so it is Borel measurable. Then,

$$S_{a,b} = \{ t \in \mathbb{R}_+ : f(t) = 0 \}$$

which is Borel measurable.

We now note that for every $t \geq 0$ the complement of the set $\mathrm{Supp}(X_t)$ is open, and therefore can be expressed as a union of open intervals (a,b)

for $a < b$. As the rational numbers are dense in the reals we can restrict to $a, b \in \mathbb{Q}$,

$$\mathbb{R} \setminus \text{Supp}(X_t) = \bigcup \{(a, b) : a < b \in \mathbb{Q}, t \in S_{a,b}\}.$$

So, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ we get the following implications

$$\begin{aligned}
(t, x) \notin \text{MSupp}(X) &\iff x \notin \text{Supp}(X_t) \\
&\iff x \in \bigcup \{(a, b) : a < b \in \mathbb{Q}, t \in S_{a,b}\} \\
&\iff (t, x) \in \bigcup \{S_{a,b} \times (a, b) : a < b \in \mathbb{Q}\},
\end{aligned}$$

which proves equality (10). $\square$

The measurability of $\text{MSupp}(X)$ follows immediately.

Lemma 4.2. *Let X be a real valued stochastic process which is right-continuous in probability. Then its marginal support is a Borel measurable subset of $\mathbb{R}_+ \times \mathbb{R}$.*

Proof. This follows immediately from equality (10). $\square$

We now show that the paths of any right-continuous process are contained in its marginal support.

Lemma 4.3. *Let X be a right-continuous real valued stochastic process. Then, with probability 1, we have*

$$\{(t, X_t) : t \in \mathbb{R}_+\} \subseteq \text{MSupp}(X).$$

Proof. We shall use proof by contradiction, so start by supposing that the statement is not true. Then, as the marginal support is Borel measurable there must exist a random time τ such that

$$\mathbb{P}\left((\tau, X_\tau) \notin \text{MSupp}(X)\right) > 0.$$

This follows from the Section Theorem (see [2] Corollary 8.5.4 or [3] Lemma 4.3). Using the notation of Lemma 4.1, it follows from equality (10) that there exists rational numbers $a < b$ such that

$$\mathbb{P}\left((\tau, X_\tau) \in S_{a,b} \times (a, b)\right) > 0.$$

Without loss of generality, we can suppose that $(\tau, X_\tau) \in S_{a,b} \times (a, b)$ whenever $\tau < \infty$. We shall show that there are only countably many possible values that τ can take.

Define the random time

$$\sigma = \inf\{t \in \mathbb{R}_+ : t \geq \tau, X_t \notin (a,b)\}$$

and the set

$$U = \{t \in \mathbb{R}_+ : \mathbb{P}(\tau < t < \sigma) > 0\}.$$

For any $t \in U$ we have $\mathbb{P}(X_t \in (a,b)) > 0$ so $(a,b) \cap \mathrm{Supp}(X_t) \neq \emptyset$. Therefore, U and $S_{a,b}$ are disjoint. As $\tau \in S_{a,b}$ whenever $\tau < \infty$ this shows that

(11) $$\mathbb{P}(\tau \in U) = 0.$$

Also, U is open. To see this, choose any $t \in U$ and a sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n \to t$. Then bounded convergence for expectations gives

$$\liminf_{n \to \infty} \mathbb{P}(\tau < t_n < \sigma) \geq \mathbb{P}(\tau < t < \sigma) > 0,$$

so $t_n \in U$ for large n. So we see that U is indeed open. Therefore U is a union of disjoint open intervals. We can write

$$U = \bigcup_{n=1}^{\infty} (u_n, v_n)$$

where $u_n, v_n \in \mathbb{R}_+ \cup \{\infty\}$ and the intervals (u_n, v_n) are disjoint.

Now, if $s < t$ are any times such that $\mathbb{P}(\tau < s < t < \sigma) > 0$ then $(s,t) \subseteq U$. So, with probability one,

$$\begin{aligned}
\rrbracket\tau, \sigma\llbracket &= \bigcup \{(s,t) : s,t \in \mathbb{Q}_+, \tau < s < t < \sigma\} \\
&\subseteq \bigcup \{(s,t) : s,t \in \mathbb{Q}_+, \mathbb{P}(\tau < s < t < \sigma) > 0\} \\
&\subseteq U.
\end{aligned}$$

Together with equation (11) this shows that whenever τ is finite then it is a left limit point of an interval in U but is not in U, with probability one. So,

$$\mathbb{P}(\tau = \infty \text{ or } \tau = u_n \text{ for some } n \in \mathbb{N}) = 1.$$

As promised, we have shown that there are only countably many possible values that τ can take. So there is a $t \in \mathbb{R}_+$ such that

$$\mathbb{P}(\tau = t) > 0.$$

Finally this gives

$$0 = \mathbb{P}(X_t \notin \mathrm{Supp}(X_t)) \geq \mathbb{P}(\tau = t) > 0,$$

which is the required contradiction.   □

In the case that X is continuous in probability (but not necessarily with continuous paths) it is easy to extend this result to show that the left limits of the process are also in the marginal support.

Lemma 4.4. *Let X be a cadlag real valued stochastic process which is left-continuous in probability. Then, with probability 1, we have*

$$\{(t, X_{t-}) : t \in \mathbb{R}_+\} \subseteq \mathrm{MSupp}(X).$$

Proof. We shall apply Lemma 4.3 to this case simply by reversing the time direction. So pick any $n \in \mathbb{N}$ and define the process

$$Y_t = X_{(n-t)_+ -}.$$

Lemma 4.3 says that with probability one, for every $t \in \mathbb{R}_+$ with $t \leq n$,

$$X_{t-} = Y_{n-t} \in \mathrm{Supp}(Y_{n-t}) = \mathrm{Supp}(X_{t-}) = \mathrm{Supp}(X_t),$$

as X is left-continuous in probability. The result now follows by letting n go to infinity. □

We are only interested in real valued processes here, but the above proofs would generalise quite easily to processes taking values in any Polish space.

We shall now move on to describe the 'almost-continuous' property of one dimensional processes in terms of the marginal support. In particular, we shall prove Lemma 1.3. First, we shall require the following useful, but very simple lemma.

Lemma 4.5. *Let A be a bounded and jointly measurable stochastic process. Then defining*

$$f : \mathbb{R}_+ \to \mathbb{R},$$
$$f(t) = \mathbb{E}\left[A_t\right],$$

f is Borel measurable. Furthermore if $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is a random time independent of A then

$$\mathbb{E}\left[1_{\{\tau < \infty\}} A_\tau\right] = \mathbb{E}\left[1_{\{\tau < \infty\}} f(\tau)\right].$$

Proof. First choose any A-measurable set S and time $s \in \mathbb{R}_+$ and define the process X by

$$X_t = 1_S 1_{\{t \geq s\}}.$$

If we set
$$g(t) = \mathbb{E}\left[X_t\right] = \mathbb{E}\left[1_S\right]1_{\{s \leq t\}}$$
then g is Borel measurable and the independence of τ and S implies
$$\mathbb{E}\left[1_{\{\tau < \infty\}}X_\tau\right] = \mathbb{E}\left[\mathbb{E}\left[1_S\right]1_{\{s \leq \tau < \infty\}}\right] = \mathbb{E}\left[1_{\{\tau < \infty\}}g(\tau)\right].$$
By the monotone class lemma this extends to any bounded A-measurable process X, and in particular applies to the case where $X = A$ and $g = f$. $\square$

A simple corollary is that two independent processes that are continuous in probability cannot jump simultaneously.

Corollary 4.6. *Let Y, Z be cadlag real valued stochastic processes such that Y is continuous in probability. Then,*
$$\mathbb{P}\left(\exists t \in \mathbb{R}_+ \ s.t. \ Y_{t-} \neq Y_t \ and \ Z_{t-} \neq Z_t\right) = 0.$$

Proof. As Z is cadlag, there exist Z-measurable random times $(\tau_n)_{n \in \mathbb{N}}$ such that $\cup_{n \in \mathbb{N}}\llbracket\tau_n\rrbracket$ contains all the jump times of Z almost-surely (see [3] Theorem 3.32). Then, Lemma 4.5 with $A_t = 1_{\{Y_t \neq Y_{t-}\}}$ gives
$$\mathbb{P}\left(\exists t \in \mathbb{R}_+ \text{ s.t. } Y_{t-} \neq Y_t \text{ and } Z_{t-} \neq Z_t\right)$$
$$\leq \sum_{n \in \mathbb{N}} \mathbb{P}\left(Y_{\tau_n-} \neq Y_{\tau_n}\right) = \sum_{n \in \mathbb{N}} \mathbb{E}\left[A_{\tau_n}\right] = 0. \qquad \square$$

We can now prove the following lemma which gives a necessary and sufficient condition for the process X to never jump past points in its marginal support. In particular, it implies that in the case where X is almost-continuous then it cannot jump past points in its marginal support. However, the converse statement is not strong enough to say that X will be almost continuous. If Y and Z are independent copies of X then the second condition below says that Y cannot jump from stricly below Z to strictly above it. Unfortunately it does not rule out the possibility that Y can approach Z from below and then jump past it (so that $Y_{t-} = Z_{t-} = Z_t < Y_t$) which would contradict almost-continuity. In order to prove Lemma 1.3 we will need to make use of the martingale property.

Lemma 4.7. *If X is a real valued process which is continuous in probability then the following are equivalent.*

1. *The set*
$$\{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : X_{t-} < x < X_t\}.$$
   is disjoint from $\mathrm{MSupp}(X)$ with probability one.

2. *Given two independent cadlag processes Y and Z, each with the same distribution as X, then*

$$\mathbb{P}\left(\exists t \in \mathbb{R}_+ \text{ s.t. } Y_{t-} < Z_t < Y_t\right) = 0$$

Proof. We first note that the second statement above is equivalent to stating that

(12) $$\mathbb{P}\left(\exists t \in \mathbb{R}_+ \text{ s.t. } Y_{t-} < a < Z_t < b < Y_t\right) = 0$$

for all real $a < b$. Let us fix any $a < b$.

As Y is cadlag, there exist Y-measurable random times $(\tau_n)_{n \in \mathbb{N}}$ such that $\cup_{n \in \mathbb{N}} [\![\tau_n]\!]$ contains all the jump times of Y almost-surely (see [3] Theorem 3.32). So equation (12) is equivalent to saying that, for every Y-measurable random time τ,

$$\mathbb{P}\left(\tau < \infty \text{ and } Y_{\tau-} < a < Z_\tau < b < Y_\tau\right) = 0.$$

Now define $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(t) = \mathbb{P}\left(a < X_t < b\right) = \mathbb{P}\left(a < Z_t < b\right).$$

Setting $A_t = 1_{\{a < Z_t < b\}}$ and

$$\sigma = \begin{cases} \tau, & \text{if } \tau < \infty \text{ and } Y_{\tau-} < a < b < Y_\tau, \\ \infty, & \text{otherwise,} \end{cases}$$

we can apply Lemma 4.5,

$$\begin{aligned}
\mathbb{P}\left(\tau < \infty \text{ and } Y_{\tau-} < a < Z_\tau < b < Y_\tau\right) &= \mathbb{E}\left[1_{\{\sigma < \infty\}} A_\sigma\right] \\
&= \mathbb{E}\left[1_{\{\sigma < \infty\}} f(\sigma)\right].
\end{aligned}$$

This term is zero if and only if $f(\sigma) = 0$ whenever $\sigma < \infty$ (a.s.). So, equation (12) is equivalent to saying that for every Y-measurable random time τ then,

$$\mathbb{P}\left(\tau < \infty, Y_{\tau-} < a < b < Y_\tau \text{ and } f(\tau) > 0\right) = 0.$$

However, as we noted above, the jump times of Y are contained in $\cup_{n \in \mathbb{N}} [\![\tau_n]\!]$, so this is equivalent to saying that

$$\begin{aligned}
&\mathbb{P}\left(\exists t \in \mathbb{R}_+ \text{ s.t. } Y_{t-} < a < b < Y_t \text{ and } f(t) > 0\right) = \\
&\mathbb{P}\left(\exists t \in \mathbb{R}_+ \text{ s.t. } X_{t-} < a < b < X_t \text{ and } f(t) > 0\right) = 0.
\end{aligned}$$

Noting that $f(t) > 0$ if and only if $(a,b) \cap \mathrm{Supp}(X_t) \neq \emptyset$, we see that equation (12) is therefore equivalent to

$$(13) \qquad \mathbb{P}\left(\exists (t,x) \in \mathrm{MSupp}(X) \text{ s.t. } X_{t-} < a < x < b < X_t\right) = 0.$$

However, saying that equation (13) is true for all real $a < b$ is equivalent to the first statement of the lemma, so the equivalence of equations (12) and (13) proves the result. $\qquad \square$

This allows us to prove Lemma 1.3 in one direction. Note that the result in this direction makes no use of the martingale property – that will be necessary when we prove the converse.

Corollary 4.8. *Let X be a real valued almost continuous process. Then the set*

$$\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : X_{t-} < x < X_t \text{ or } X_t < x < X_{t-}\}$$

is disjoint from the marginal support of X, with probability one.

Proof. Let Y and Z be independent cadlag processes with the same distribution as X. By Corollary 4.6, the jump times of Y and Z are disjoint sets (restricting to a set of probability one).

Now suppose that, with positive probability, $Y_{t-} < Z_t < Y_t$ for some t. Then by the continuity of Z at t and the right-continuity of Y, there exist times $u, v \in \mathbb{Q}_+$ such that $u < t < v$ and $Y_u < Z_u$, $Y_v > Z_v$ and $Y_s \neq Z_s$ for every $s \in (u,v)$. As this contradicts Definition 1.1 of almost-continuity, we see that the second statement of Lemma 4.7 is true. Therefore, by Lemma 4.7, the set

$$\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : X_{t-} < x < X_t\}$$

is almost-surely disjoint from $\mathrm{MSupp}(X)$. Similarly, applying the same argument to $-X$ gives the result. $\qquad \square$

We still need to show that Lemma 1.3 is true in the opposite direction, for which we need the following result. This is effectively saying that the process Y cannot approach Z from below without either touching or jumping past it. Note that it also says that if the processes are adapted, then the stopping time T is previsible on the set $Y_T \neq Z_T$.

Lemma 4.9. *Let X be a cadlag real valued process which is continuous in probability, and such that the set*

$$\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : X_{t-} < x < X_t \text{ or } X_t < x < X_{t-}\}$$

is disjoint from $\mathrm{MSupp}(X)$ *with probability one.*

Also, let Y *and* Z *be independent cadlag processes each with the same distribution as* X. *For any* $s \in \mathbb{R}_+$ *let* T *be the random time*

$$T = \begin{cases} \inf\{t \in \mathbb{R}_+ : t \geq s, Y_t \geq Z_t\}, & \text{if } Y_s < Z_s, \\ \infty, & \text{otherwise,} \end{cases}$$

and $(T_n)_{n \in \mathbb{N}}$ *be the random times*

$$T_n = \begin{cases} \inf\{t \in \mathbb{R}_+ : t \geq s, Y_t + 1/n \geq Z_t\}, & \text{if } Y_s < Z_s, \\ \infty, & \text{otherwise.} \end{cases}$$

Then $T_n \uparrow T$ *as* $n \to \infty$ *(a.s.). Also,* $T_n < T$ *whenever* $T < \infty$ *and* $Y_T \neq Z_T$ *(a.s.).*

Proof. First, it is clear from the definitions that $T_n \leq T$ for each n and that T_n is an increasing sequence. So we can define a random time S by

$$S = \lim_{n \to \infty} T_n.$$

We need to show that $S = T$, for which it is enough to prove $Y_S \geq Z_S$ whenever $S < \infty$. We start by showing that Y cannot have a negative jump at time S. Choose any $m \in \mathbb{N}$ and let A be the set

$$A = \{\forall n \in \mathbb{N}, \ T_n < S < \infty \text{ and } Y_S \leq Y_{S-} - 1/m\}.$$

Note that in A we necessarily have

$$|Z_{S-} - Y_{S-}| = \lim_{n \to \infty} |Z_{T_n} - Y_{T_n}| \leq \lim_{n \to \infty} 1/n = 0,$$

so $Y_{S-} = Z_{S-}$. Define the function

$$\begin{aligned} f &: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \\ f(t, x) &= \mathbb{P}\left(x - 1/m < X_t < x\right), \end{aligned}$$

so $f(t, x) = 0$ if and only if $(x - 1/m, x) \cap \mathrm{Supp}(X_t) = \emptyset$.

The conditions of the lemma say that that Y cannot jump past any points of the marginal support of X. However, on the set A, we have $(Y_{S-} - 1/m, Y_{S-}) \subseteq (Y_S, Y_{S-})$, so

$$f(S, Z_{S-}) = f(S, Y_{S-}) = 0 \text{ (a.s.)}.$$

Restricting to A we have $Y_{S-} = Z_{S-}$ and $Y_{t-} < Z_{t-}$ for $s < t < S$. So with probability one there exists a $u \in \mathbb{Q}_+$ with $u < S$ and such that

$$Z_{t-} > Y_{t-} > Z_{t-} - 1/m$$

for every $t \in [u, S)$. As the paths of Y_- lie in the marginal support of X (by Lemma 4.4) this implies that, restricting to A, $f(t, Z_{t-}) > 0$ for every $t \in [u, S)$ (almost surely).

So, if for every $u \in \mathbb{Q}_+$, we define the random time

$$R^u = \inf \{t \in \mathbb{R}_+ : t \geq u, f(t, Z_{t-}) = 0\}$$

then $R^u = S$ for some $u \in \mathbb{Q}_+$. Note that the Debut Theorem ([3] Theorem 4.2) says that R^u is Z-measurable. Therefore,

$$\mathbb{P}(A) \leq \sum_{u \in \mathbb{Q}_+} \mathbb{P}(A \cap \{S = R^u\}) \leq \sum_{u \in \mathbb{Q}_+} \mathbb{P}(R^u < \infty, \; Y_{R^u-} \neq Y_{R^u}).$$

As the random times R^u are Z-measurable, they are independent of Y. So Lemma 4.5 with $A_t = 1_{\{Y_t \neq Y_{t-}\}}$ gives

$$\mathbb{P}(R^u < \infty, \; Y_{R^u-} \neq Y_{R^u}) = 0.$$

So $\mathbb{P}(A) = 0$. More explicitly,

$$\mathbb{P}(\forall n \in \mathbb{N}, \; T_n < S < \infty \text{ and } Y_S \leq Y_{S-} - 1/m) = 0$$

Letting m go to infinity tells us that

$$(\forall n \in \mathbb{N}, \; T_n < S < \infty) \Rightarrow Y_S \geq Y_{S-} \text{ (a.s.)}.$$

Similarly, replacing Y by $-Z$ and Z by $-Y$ in the above argument gives

$$(\forall n \in \mathbb{N}, \; T_n < S < \infty) \Rightarrow Z_S \leq Z_{S-} \text{ (a.s.)}.$$

However, we have $Y_{T_n} + 1/n \geq Z_{T_n}$. Therefore, if $T_n < S < \infty$ for every n then $Y_{S-} \geq Z_{S-}$. So, we get

$$(\forall n \in \mathbb{N}, \; T_n < S < \infty) \Rightarrow Y_S \geq Z_S \text{ (a.s.)}.$$

On the other hand, if $T_n = S < \infty$ for any n, then

$$Y_S = Y_{T_m} \geq Z_{T_m} - 1/m = Z_S - 1/m$$

for every $m \geq n$, and therefore $Y_S \geq Z_S$. We have shown that $Y_S \geq Z_S$ whenever $S < \infty$. So $T = S$ and $T_n \to T$.

Finally, suppose that $T_n = T < \infty$ for large n. Then, $Y_T \geq Z_T$ and $Y_{T-} \leq Z_{T-} - 1/n$. Corollary 4.6 says that $Y_{T-} = Y_T$ or $Z_{T-} = Z_T$, so one of the following inequalities must be true,

$$Z_T \leq Y_T = Y_{T-} < Z_{T-} \text{ or } Y_{T-} < Z_{T-} = Z_T \leq Y_T.$$

However Lemma 4.7 says that we cannot have $Y_{T-} < Z_T < Y_T$. Similarly, by replacing Y with $-Z$ and Z with $-Y$ then we cannot have $Z_T < Y_T < Z_{T-}$. So, we must have $Y_T = Z_T$. Therefore, if $Y_T \neq Z_T$ then $T_n < T$ for every n. □

Finally, we use the above result to prove Lemma 1.3.

Proof of Lemma 1.3. First, if X is almost-continuous then the result follows from Corollary 4.8. It only remains to show the converse.

So suppose that the set given by equation (3) is almost surely disjoint from $\mathrm{MSupp}(X)$. Let Y and Z be independent cadlag processes each with the same distribution as X, and pick any $s \in \mathbb{R}_+$. Also, let T and $(T_n)_{n \in \mathbb{N}}$ be the stopping times defined by Lemma 4.9. Picking any $t > s$ then the martingale property gives,

$$\mathbb{E}\left[1_{\{T_n \leq t\}}(Y_{t \wedge T} - Z_{t \wedge T})\right] = \mathbb{E}\left[1_{\{T_n \leq t\}}(Y_{T_n} - Z_{T_n})\right].$$

Noting that $Y_{T_n} < Z_{T_n}$ whenever $T_n < T$, and, by Lemma 4.9, $Y_{T_n} = Z_{T_n}$ whenever $T_n = T < \infty$, we get

$$\mathbb{E}\left[1_{\{T \leq t\}}(Y_T - Z_T)\right] = \lim_{n \to \infty} \mathbb{E}\left[1_{\{T_n \leq t\}}(Y_{T_n} - Z_{T_n})\right] \leq 0.$$

As $Y_T \geq Z_T$ this shows that $Y_T = Z_T$ whenever $T \leq t$. However, if $Y_s < Z_s$ and $Y_t > Z_t$ then $T < t$, so

$$\mathbb{P}\left(Y_s < Z_s, Y_t > Z_t \text{ and } Y_u \neq Z_u \text{ for every } u \in (s,t)\right) = 0. \qquad \square$$

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